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Gravity on codimension 2 brane worlds

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Abstract: We compute the matching conditions for a general thick codimension 2 brane, a necessary previous step towards the investigation of gravitational phenomena in codimension 2 braneworlds. We show that, provided the brane is weakly curved, they are specified by the integral in the extra dimensions of the brane energy-momentum, independently of its detailed internal structure. These general matching conditions can then be used as boundary conditions for the bulk solution. By evaluating Einstein equations at the brane boundary we are able to write an evolution equation for the induced metric on the brane depending only on physical brane parameters and the bulk energy-momentum tensor. We particularise to a cosmological metric and show that a realistic cosmology can be obtained in the simplest case of having just a non-zero cosmological constant in the bulk. We point out several parallelisms between this case and the codimension 1 brane worlds in an AdS space.

Keywords: Field Theories in Higher Dimensions, p-branes, Classical Theories of Gravity.

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1. Introduction

New cosmological observations seem to imply that the expansion of our universe is currently accelerating [1], driven by a dominant component of the energy-momentum tensor (EMT) with an equation-of-state parameter w close to -1 (the so called dark energy). The observations have made the cosmological constant problem a very pressing one: to the traditional puzzle of an (almost) vanishing vacuum energy now cosmologists (and particle physicists) wonder why is its magnitude comparable to the matter energy density *today*. Recent analysis of the data [2] point to an even more bizarre situation: the best fit to observations agrees with a dark energy equation of state with $\omega < -1$. So the family of problems associated with the vacuum energy (the cosmological constant problem and its smaller cousin, the coincidence problem) could grow in the near future with a new member: why is the vacuum energy *growing* in time? The problem with this possibility is that it is not easily accommodated in generally covariant theories as long as the matter EMT satisfies the usual energy conditions [3]. In the same fashion, other existing observations also suggest modified gravitational dynamics.[1]

Weinberg's theorem [6] shows that standard approaches (by which we mean 4D field theories based on General Relativity) to the cosmological constant problem are very likely to fail and therefore more exotic ones should be tried. In particular having more than four dimensions in a Kaluza-Klein fashion does not seem to improve the situation, since the extra dimensions are small and gravity is effectively four dimensional below some scale. In this effective theory one will face the same problems as in any 4D theory. Thus, within this class of theories, anthropic considerations seem at the moment the only framework capable of explaining some of the large scale properties of our universe [7]. Brane-world gravity, on the other hand, does not belong to this class of theories, since it is not guaranteed that the low energy description of gravity can be obtained from a generally covariant four-dimensional

[1] One can mention the flatness of galaxy rotation curves, that can be explained using the dark matter hypothesis but that can also be regarded as pointing towards modified gravitational dynamics [4], or the more nearby measured Pioneer 10/11 anomalies [5].

lagrangian, i.e., there is *not* necessarily a four dimensional description of the gravitational sector. In these models one assumes that the Standard Model fields are confined to some submanifold of the whole space-time. One can think on the Standard Model fields as the zero modes of topological defects of higher dimensional field theories [8, 9] or the gauge theories living on the world-volume of the string theory D-branes [10]. Fermionic fields and gauge interactions can be in this way clearly lower dimensional but to elucidate if the gravitational interactions can be well approximated by 4D Einstein gravity is not so obvious. The hope is to find in this context a theory that shares the good conceptual advantages of 4D General Relativity (gravity as geometry, background independence, ...) but can yield a realistic but non-standard cosmology (or gravitational dynamics).

The cosmology of codimension one braneworlds is quite well understood. It is possible to recover something close to standard cosmology with corrections that take the form of "dark radiation" plus terms involving the matter energy-momentum tensor squared [11, 12]. Although the situation is not as good in the understanding of codimension two brane worlds, great progress has been made recently in their investigation. The fact that one can find solutions for a flat brane in a given setup for any value of its tension has encouraged many authors to try and attack the cosmological constant problem in such scenario [13]–[18]. The effect of the brane tension in these models is simply to produce a deficit angle in the transverse space, without further implications for its induced metric. The two dimensional space transverse to the brane acquires locally the geometry of a cone, with the brane situated on its tip. However one problem of codimension two brane-worlds is that with a deficit angle one can only generate a two dimensional delta function in the Einstein tensor that is proportional to the brane induced metric. This means that one can only find nonsingular solutions if the brane EMT is proportional to its induced metric, i.e. it is pure tension [15, 16]. Thus the solutions found in [13] cannot be extended to a general brane EMT in the thin brane limit. This limit is indeed singular for a general brane and, as such, makes all the arguments about the nature of gravity (and self-tuning) in codimension two braneworlds in Einstein gravity dodgy when working with δ-like branes. To make things worse, the very Einstein equations imply that in the case of an infinitesimally thin pure tension brane, the deficit angle is space and time independent. This situation is very similar to the cosmic string models studied in 4D (see [19] for a rigorous treatment of codimension one and two sources in 4D General Relativity). It is therefore not sensible to ask questions like what happens if there is a sudden phase transition that changes the tension of the brane as such thing is not allowed by the equations of motion in the thin brane limit. In other words, self-tuning has to be formulated as a dynamical process and delta-like codimension two branes do not allow to do that. In [16] a possible way out of this situation was proposed by adding the Gauss-Bonnet term to the Einstein-Hilbert lagrangian. In this case, the thin brane limit is well defined and one finds the remarkable result that four dimensional Einstein gravity is recovered as the dynamics for the induced metric on the infinitesimally thin brane. (Similar conclusions have been obtained in [20] at the linearised level.) See also [21, 22] for a different approach to codimension 2 braneworlds in Gauss-Bonnet gravity.

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Motivated by the previous considerations, we abandon in this paper the thin brane idealization, and compute the "matching" conditions for a general thick codimension two brane in Einstein gravity.[2] By matching conditions we mean equations that relate the values of the first derivatives of the metric (with respect to the orthogonal coordinate, r) at the brane boundary with the brane EMT. In fact, since we are dealing with a thick defect (with a singular thin limit), it is not clear that one should be able to do this without knowing all the precise small scale structure of the brane. If we need this information in order to find the solution, the matching condition approach would be of no use, since one should solve the equations independently for different microscopic brane models, and one could obtain different results for different models even if the total energy-momentum carried by the brane is the same. Gravitational physics would then in general depend on the ultraviolet details of our theory and therefore model independent assertions would be difficult to make. We will see, however, that one can obtain this set of equations depending only on the integral of the brane EMT along the extra dimensions as long as the parallel derivatives (i.e. with respect to the brane coordinates) of the metric, and in particular the Ricci tensor of the brane induced metric, are small enough: in this case our matching conditions do not depend on the inner structure of the brane. How small is "small enough" will be made clear in the next section but one can argue that this situation is quite general in the sense that it is natural that the presence of the brane induces much larger gradients in the radial (transverse) direction that in the longitudinal ones. In particular this is clearly the case if we are interested in late time cosmology. Of course that does not mean that nothing can be said about very early cosmological times or other situations with strong gravity effects, but one should keep in mind when dealing with such situations that our matching conditions imply certain approximations that break down when the 4D curvature is very large. Once we consider a thick brane, departure from pure tension is allowed and the deficit angle can develop a space-time dependence, thus questions about the self-tuning behaviour of the system are again legitimate.

In the next section we will carefully explain our assumptions and approximations and we will obtain the equations that relate the *total* brane energy-momentum with the deformation of spacetime it causes, the so called matching conditions. We will specialize these equations to the cosmological case in the third section. Evaluating Einstein equations just outside the brane and using the matching conditions we will be able to obtain the equations that govern the cosmological evolution of this braneworld. In this respect, we follow a procedure completely analogous to that of the codimension one case [12].

2. Matching conditions for a codimension 2 brane

In this section we will try to answer the following questions: what is the effect on space-time of an energy-momentum distribution that can be interpreted as a codimension 2 defect in six-dimensional Einstein gravity? Is there a way to characterise this effect without knowing

[2]We do not introduce the Gauss-Bonnet term now because once one considers a thick brane the Gauss-Bonnet contributions, although crucial in the thin limit for obtaining a regular geometry, will be subleading unless the Gauss-Bonnet coupling is very large or the brane is extremely thin.

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all the precise small-scale structure of the brane? We will see that the answer to the second question is yes, provided certain conditions are met, and we will also provide a (partial) answer to the first question. The needed conditions have the interpretation of requiring a weakly curved brane.

In this section we will consider the following quite general ansatz for the metric,

$$ds^2 = g_{\mu\nu}(x,r)dx^\mu dx^\nu - dr^2 - L(x,r)^2 d\theta^2\,, \tag{2.1}$$

where, as usual, x^μ denote four non-compact dimensions (including a time-like one), $\mu = 0, \ldots, 3$, whereas r, θ denote the radial and angular coordinates of the (compact or not) two extra dimensions. This means that in particular the following boundary conditions hold (in order to avoid singularities) at $r = 0$

$$L(x,0) = 0\,, \qquad L'(x,0) = 1\,, \qquad \partial_r g_{\mu\nu}(x,0) = 0\,, \tag{2.2}$$

where a prime denotes derivative with respect to r. We have assumed a rotational symmetry around the codimension 2 submanifold defined by the condition $r = 0$. The metric is determined by the Einstein equation, that can be written as

$$M_*^4 R^M_N = T^M_N - \frac{1}{4}\delta^M_N T\,, \tag{2.3}$$

where M_* is the 6D fundamental mass, T_{MN} is the EMT and $T \equiv T^M_M$ its trace. The brane will be a cylindrically symmetric extended object that fills the region with $r < \epsilon$. Since we are trying to deal here with a general situation, and we do not have a particular microscopic theory for the brane, we cannot provide a precise definition of the brane width parameter, ϵ, but in a given particular model it should not be too hard to provide a strict definition of it. In any case, the same results should be obtained taking a different definition of the brane width (i.e. a different splitting into brane-bulk of the total 6D EMT), as long as we are considering the same energy-momentum distribution.

The presence of the brane induces a strong r-dependence of the curvature tensor that should be reflected on large r-derivatives of the metric. It is natural then to assume that the brane has the effect of producing mainly non-zero r-derivatives of the metric, and these derivatives are the relevant terms in the Einstein equations when looking for a solution. So, given the boundary conditions at $r = 0$, eq. (2.2), we would like to obtain the values for the first derivatives of the metric at $r = \epsilon$ in terms of the brane EMT.[3] For doing this we follow the standard procedure of integrating the equations of motion in the region with $r < \epsilon$. Notice that if we are to find a result that is independent of the inner structure of the brane, the dominant terms of the integral should be total r-derivatives. If this is the case the value of the integral depends only on the value of some functions on the boundary, $r = \epsilon$ (and the origin, $r = 0$), and not on the precise solution inside the brane.

The set of equations we will have to deal with, for the metric at question, will be given next. We will offer this equations in a form that makes transparent which terms can be integrated exactly and which ones should be neglected when dealing with the matching.

[3] We call brane EMT to $T_{MN}(r < \epsilon)$, including possible contributions from the bulk EMT inside the extension of the brane.

The $\mu\nu$ components of the Ricci tensor can be written as,

$$\sqrt{g}LR^{\mu}_{\nu} = \frac{1}{2}[\sqrt{g}LK^{\mu}_{\nu}]' + \sqrt{g}LR^{\mu}_{\nu}(g) - \sqrt{g}\nabla^{\mu}\nabla_{\nu}L\,, \tag{2.4}$$

where $K_{\mu\nu} \equiv \partial_r g_{\mu\nu}$ (we will also use $K \equiv K^{\mu}_{\mu}$), ∇_μ denote four-dimensional (i.e. with respect to the metric $g_{\mu\nu}$) covariant derivatives and $R_{\mu\nu}(g)$ is the Ricci tensor for the four-dimensional metric $g_{\mu\nu}$. The $\theta\theta$ component of Ricci tensor reads

$$\sqrt{g}LR^{\theta}_{\theta} = [\sqrt{g}L']' - \sqrt{g}\nabla^{\rho}\nabla_{\rho}L\,. \tag{2.5}$$

We see that these two equations can be written as a total r derivative plus terms involving only derivatives with respect to the longitudinal coordinates (x^μ). As we will see in a moment, they will determine the matching conditions. As for the other two non-vanishing components of Einstein equations for our metric, the rr and μr components of the Ricci tensor read, respectively,

$$R^{r}_{r} = \frac{L''}{L} + \frac{1}{2}K' + \frac{1}{4}K^{\rho}_{\sigma}K^{\sigma}_{\rho}\,, \tag{2.6}$$

and

$$R_{\mu r} = -\frac{\partial_\mu L'}{L} + \frac{1}{2}\frac{\partial_\nu L}{L}K^{\nu}_{\mu} + \frac{1}{2}\nabla^{\nu}(K_{\mu\nu} - g_{\mu\nu}K)\,. \tag{2.7}$$

We can now integrate the $\mu\nu$ and $\theta\theta$ components of the Ricci tensor in the region $0 \le r \le \epsilon$ (the integration in θ is trivial) to find the desired matching conditions. We start with the $\mu\nu$ components, eq. (2.4). Integrating this equation and neglecting the terms that do not have r-derivatives[4] one gets

$$\frac{2\pi}{\sqrt{g}|_\epsilon}\int_0^\epsilon dr\,\sqrt{g}LR^{\mu}_{\nu} \simeq \pi\frac{K^{\mu}_{\nu}|_\epsilon}{M_b} \simeq \frac{1}{M_*^4\sqrt{g}|_\epsilon}\int_0^{2\pi} d\theta \int_0^\epsilon dr\,\sqrt{g}L\left[T^{\mu}_{\nu} - \frac{1}{4}\delta^{\mu}_{\nu}T\right] \equiv \frac{\hat{T}^{\mu}_{\nu} - \frac{1}{4}\delta^{\mu}_{\nu}\hat{T}}{M_*^4}\,, \tag{2.8}$$

where we have defined $L(x,\epsilon) \equiv 1/M_b$ (notice that $1/M_b \sim \epsilon$) and it is understood in here and in the following that the subscript $|_p$ means that the corresponding function is evaluated at $r = p$. We have also defined the 4D brane EMT, $\hat{T}^{N}_{M}$, as the integration of the full 6D EMT in the region with $r < \epsilon$:

$$\hat{T}^{N}_{M} \equiv \frac{1}{\sqrt{g}|_\epsilon}\int_0^{2\pi} d\theta \int_0^\epsilon dr\,\sqrt{g}LT^{N}_{M}\,, \tag{2.9}$$

and $\hat{T} \equiv \hat{T}^{M}_{M}$ as its trace.

Now we can repeat the procedure with the $\theta\theta$ equation. Performing the corresponding integration and neglecting again the terms with only longitudinal derivatives we get

$$\frac{2\pi}{\sqrt{g}|_\epsilon}\int_0^\epsilon dr\sqrt{g}LR^{\theta}_{\theta} \simeq 2\pi\left(\beta - \frac{\sqrt{g}|_0}{\sqrt{g}|_\epsilon}\right) \simeq \frac{1}{M_*^4\sqrt{g}|_\epsilon}\int_0^{2\pi} d\theta \int_0^\epsilon dr\,\sqrt{g}L\left[T^{\theta}_{\theta} - \frac{1}{4}T\right] \equiv \frac{\hat{T}^{\theta}_{\theta} - \frac{1}{4}\hat{T}}{M_*^4}\,, \tag{2.10}$$

[4]For the cosmological case we will estimate $R^{\mu}_{\nu}(g)$ and the other neglected terms of eqs. (2.4), (2.5) that involve μ-derivatives, but to obtain the cosmological evolution we need first the matching conditions. We will see at the end that these terms are indeed negligible (with respect to the terms we have kept) in the cosmological solutions we consider. This is not surprising since once one imposes the constraints of having a realistic late-time cosmology the 4D curvature or μ-dependence of the solution have to be extremely small.

where we have defined $\beta(x) \equiv L'(x, \epsilon)$. This equation, together with eq. (2.8), determine the exterior space-time geometry associated with a particular energy-momentum stored in our codimension two defect. It is apparent that the $\theta\theta$ equation fixes the deficit angle in the transverse space, while the $\mu\nu$ equations have the clear interpretation of requiring a non-zero extrinsic curvature at $r = \epsilon$ unless $\hat{T}^\mu_\nu - \frac{1}{4}\delta^\mu_\nu \hat{T} = 0$. The trace of this quantity is referred to as the Tolman mass in 4D cosmic string literature (see e.g. [23]), and it is zero for a pure tension brane. For obtaining these equations we have only neglected the terms that do not involve r-derivatives in the integrals.

Notice that the $\mu\nu$ matching conditions are very similar to those of a codimension one brane, but because $L(x, 0) = 0$ we cannot satisfy this equation with a finite K^ν_μ in the thin brane limit ($M_b \to \infty$) in general, in contrast with the codimension one case that has a well defined thin limit [19, 24]. In fact it is instructive to compare the codimension two case with the more familiar codimension one case in more detail, and point out their similarities and differences. For a 5D metric like

$$ds^2 = g_{\mu\nu}(x, r)dx^\mu dx^\nu - dr^2 \,, \tag{2.11}$$

we can write the $\mu\nu$ components of the Ricci tensor as

$$\sqrt{g} R^\mu_\nu = \frac{1}{2}\left[\sqrt{g} K^\mu_\nu\right]' + \sqrt{g} R^\mu_\nu(g) \,, \tag{2.12}$$

where, as before, $K_{\mu\nu} \equiv \partial_r g_{\mu\nu}$ and $R^\mu_\nu(g)$ is the Ricci tensor for the 4D metric $g_{\mu\nu}$. We can integrate now this equation from $r = -\epsilon$ to $r = \epsilon$, a region where we assume that some energy-momentum density is localized. We get then

$$\frac{1}{\sqrt{g}|_0} \int_{-\epsilon}^{\epsilon} \sqrt{g} R^\mu_\nu = \frac{1}{2\sqrt{g}|_0}\left[\sqrt{g} K^\mu_\nu\right]^\epsilon_{-\epsilon} + \frac{1}{\sqrt{g}|_0} \int_{-\epsilon}^{\epsilon} \sqrt{g} R^\mu_\nu(g) = \frac{\hat{T}^\mu_\nu - \frac{1}{3}\delta^\mu_\nu \hat{T}}{M_*^3}. \tag{2.13}$$

M_* is now the 5D fundamental mass, $\hat{T}^N_M$ is again the integration of the full 5D EMT in the $(-\epsilon, \epsilon)$ region,

$$\hat{T}^N_M \equiv \frac{1}{\sqrt{g}|_0} \int_{-\epsilon}^{\epsilon} dr \, \sqrt{g} T^N_M \,, \tag{2.14}$$

and $\hat{T}$ its trace. It is clear in this case that we can take the limit $\epsilon \to 0$ keeping $\hat{T}_{\mu\nu}$, K^μ_ν and $R^\mu_\nu(g)$ finite if we accept a discontinuity in the extrinsic curvature (notice that we still have $\sqrt{g}|_{\pm\epsilon} \to \sqrt{g}|_0$ in this limit). We obtain in this way the so called Israel matching conditions [24]

$$\frac{1}{2} M_*^3 \left[K^\mu_\nu\right]^{0^+}_{0^-} = \hat{T}^\mu_\nu - \frac{1}{3}\delta^\mu_\nu \hat{T} \,, \tag{2.15}$$

since only the extrinsic curvature contribution survives in the thin limit. One could think that the analogy between the thick codimension two brane and the codimension one case is not surprising since, once the brane has been given a certain width, the wall defining the brane is indeed a codimension one object. But there is some information in our codimension two matching conditions showing that our system is different from a codimension one brane.[5] First, we have an extra dimensional matching condition, the deficit angle contribution eq. (2.10). As we have mentioned, in the case of a codimension two pure tension brane with $\hat{T}^r_r = \hat{T}^\theta_\theta = 0$, this contribution absorbs completely the effect of the brane on the background, allowing us to keep zero extrinsic curvature. This is not the case for the codimension one brane, since the right hand side of eq. (2.13) does not vanish for a pure tension brane. This difference is what makes codimension two braneworlds attractive as a possible solution to the cosmological constant problem. Also, in the codimension two case we cannot take the thin limit because $\sqrt{G} = L\sqrt{g}$ vanishes at $r = 0$, and then the left hand side of eq. (2.8) goes to zero as $\epsilon \to 0$ (even allowing for discontinuities in the first derivatives of the metric) if we insist on keeping K^ν_μ finite. It is then natural to expect that, keeping ϵ finite but small, in a codimension two brane-world situation the extrinsic curvature contribution is still the main contribution to the integral (2.8). Thus, we can safely neglect the integration of the terms involving only μ-derivatives in most situations, most notably if we are interested in late cosmological times for which 4D curvatures are extremely small.

Another interesting feature of our matching conditions is that we can now use them in the μr equation evaluated at $r = \epsilon$ to obtain an energy-momentum conservation equation for the brane EMT

$$M_*^4 L G_{\mu r} = \frac{1}{2\pi}\left(\nabla^\nu \hat{T}_{\mu\nu} + \nabla_\mu \hat{T}^r_r\right) + \frac{\hat{T}^r_r + \hat{T}^\theta_\theta}{2\pi}\frac{\partial_\mu M_b}{M_b} - M_*^4 \partial_\mu\left(\frac{\sqrt{g}|_0}{\sqrt{g}|_\epsilon}\right) = \frac{T^{\text{bulk}}_{\mu r}|_\epsilon}{M_b}. \tag{2.16}$$

The μr component of the bulk EMT determines the flow of energy-momentum from the brane into the bulk. However, even when that term is zero, there can be an exchange of energy-momentum between the longitudinal and the transverse directions (on the brane) correlated with a possible space-time dependence of the brane width that could be interpreted by 4D observers as apparent violations of the conservation of energy-momentum.

Up to now we have performed an integration of the 6D Einstein equations in the space-time region filled by the brane in order to obtain the matching conditions. We have identified a set of terms in the equations that can be integrated in a model independent way, and we have seen that these terms are indeed the dominant ones provided the μ-dependence of the solution is small. In this way we have obtained eqs. (2.8), (2.10) that relate the first (transverse) derivatives of the metric just outside the brane with the integrated brane EMT. We are taking $g_{\mu\nu}|_\epsilon$ as our "induced metric" for the defect, since we are evaluating most functions in $r = \epsilon$ when dealing with the matching. In fact we see that all the functions appearing in the matching are evaluated at the brane boundary except for the ratio $\sqrt{g}|_0/\sqrt{g}|_\epsilon$ appearing in the $\theta\theta$ matching condition, eq. (2.10) and in the EMT conservation equation, eq. (2.16). In order to avoid making reference to functions evaluated *inside* the brane when dealing with the matching we will consider that this ratio can be approximated by one in this equation. For this we just need that

$$\left|\hat{T}^r_r + \hat{T}^\theta_\theta\right| \ll \left|\hat{T}^\mu_\mu\right|. \tag{2.17}$$

[5] This difference comes ultimately from the boundary conditions we are imposing at $r = 0$, eqs. (2.2). If we were imposing instead the boundary conditions $L'(x,0) = 0$ and $L(x,0)$ different from zero, we would be describing a 4-brane with a compact dimension (θ) and a Z_2 symmetry at $r = 0$.

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This is because we can put a bound on the difference $\sqrt{g}|_\epsilon - \sqrt{g}|_0$ as

$$\sqrt{g}|_\epsilon - \sqrt{g}|_0 \leq \sqrt{g}|_0 \frac{K|_{\max}\epsilon}{2} \sim \sqrt{g}|_0 \frac{K|_\epsilon}{2M_b}\,, \tag{2.18}$$

where $K|_{\max}$ is the maximum value of the function for $r \leq \epsilon$ and we have used $\partial_r\sqrt{g} = \sqrt{g}K/2$. Requiring $\frac{\sqrt{g}|_0}{\sqrt{g}|_\epsilon} - 1 \ll \beta - 1$ (so that the ratio can be approximated by 1 *in the $\theta\theta$ matching equation*) and using the matching conditions, eqs. (2.8), (2.10), we arrive at the mentioned requirement, eq. (2.17). When this condition holds, we could speak of a quasi-pure-tension-brane, and the extrinsic curvature (times the brane width) is negligible with respect to the deficit angle. This allows us to approximate the ratio of the metric determinants in eq. (2.10) by one. We quote here the actual matching conditions again, assuming such condition holds

$$2\pi(1-\beta) \simeq \frac{1}{M_*^4}\left(\frac{1}{4}\hat{T} - \hat{T}^\theta_\theta\right) \simeq \frac{1}{4M_*^4}\hat{T}^\mu_\mu\,, \tag{2.19}$$

$$K^\nu_\mu|_\epsilon \simeq \frac{M_b}{\pi M_*^4}\left[\hat{T}^\nu_\mu - \frac{1}{4}\hat{T}\delta^\nu_\mu\right]. \tag{2.20}$$

We would like to emphasize the generality of these matching conditions. They apply to *any* codimension two brane, provided we can neglect the longitudinal μ-derivatives when compared with the transverse r-derivatives in the solution *inside* the brane, and our condition eq. (2.17) holds. For instance, it is now straightforward to interpret several known solutions with naked codimension two singularities as being sourced by a codimension two object with certain energy-momentum. Consider as an example metrics that near $r = 0$ can be approximated as

$$g_{\mu\nu} \simeq \kappa_1 r^{\alpha_1}\eta_{\mu\nu} + \cdots\,, \tag{2.21}$$

$$L \simeq \kappa_2 r^{\alpha_2} + \cdots\,. \tag{2.22}$$

Our matching conditions then imply

$$1 - \frac{\kappa_2\alpha_2}{M_b^{\alpha_2-1}} \simeq \frac{T_0}{2\pi M_*^4}\,, \tag{2.23}$$

$$\alpha_1 = -\frac{1}{4\pi M_*^4}\left(\hat{T}^\theta_\theta + \hat{T}^r_r\right), \tag{2.24}$$

where we have taken $\hat{T}^\nu_\mu = T_0\ \delta^\nu_\mu$ and we are assuming that eq. (2.17) holds. We see how the required energy-momentum for the defect sourcing these solutions depends on the brane width. The brane width acts as a cut-off for the curvature, and gets rid of the singularity once one considers a thick defect: remember that these relations have been obtained matching a regular geometry at $r = 0$ (that implies eqs.(2.2)) with the exterior geometry given by eqs.(2.21), (2.22). It is interesting to point out that in the thin brane limit ($M_b \to \infty$), the brane tension diverges if $\alpha_2 < 1$ while it goes to zero if $\alpha_2 > 1$ (in this latter case we can not satisfy our assumption in the thin limit, eq. (2.17), and the first matching condition above would have some corrections). We can recognize the only parameters with a well defined thin brane limit yielding a finite brane energy-momentum ($\alpha_1 = 0$, $\alpha_2 = 1$) as a purely conical geometry. In particular the singular solutions of 6D supergravity found in [14], can be cast in the form given by eqs. (2.21), (2.22) and interpreted as being sourced by a codimension 2 defect with an energy momentum tensor given by the formulae above.

Also, we can now match an exterior AdS geometry with a regular geometry on $r = 0$ and check what kind of energy-momentum distribution supports such spacetimes. This kind of exterior geometry has been obtained as the spacetime produced by the Nielsen-Olesen vortex of the abelian Higgs model in 6D[6] [26] which, interestingly enough, localizes not only gravity but also gauge interactions at the vortex core [27] (see also [28] for the non-abelian case). Applying our matching conditions to an exterior geometry like (AdS_6 space with a compact dimension)

$$ds^2 = e^{\pm kr}\eta_{\mu\nu}dx^\mu dx^\nu - L_0^2 e^{\pm kr}d\theta^2 - dr^2\,, \tag{2.25}$$

where $k = \sqrt{-\frac{5\Lambda}{2M_*^4}}$, and Λ is the bulk cosmological constant, we obtain the required brane EMT as

$$1 \mp \frac{L_0 k}{2} e^{\pm \frac{k}{2M_b}} \simeq \frac{T_0}{2\pi M_*^4}\,, \tag{2.26}$$

$$\pm\frac{k}{M_b} = -\frac{1}{4\pi M_*^4}\left(\hat{T}_\theta^\theta + \hat{T}_r^r\right), \tag{2.27}$$

where we have taken again $\hat{T}_\mu^\nu = T_0\ \delta_\mu^\nu$ and we are assuming that eq. (2.17) holds. Notice that if we choose the minus sign for $\pm k$, so the volume of the spacetime is finite, we need $T_0 > 2\pi M_*^4$. This condition does not mean that the curvatures are in the solution bigger than the fundamental mass, since we expect that $\hat{T}_\mu^\nu \sim T_\mu^\nu \epsilon^2 \sim T_\mu^\nu/M_b^2$. For low values of M_b we can still have the 4D brane EMT ($\hat{T}_\mu^\nu$) of the order of M_*^4 or bigger while the 6D one (T_μ^ν) is hierarchically smaller and the curvatures in the full 6D solution are under control, but one can not keep curvatures under control in the thin brane limit (see also the discussion in [26]). However, if we want to restrict ourselves to weakly gravitating branes we should choose the plus sign, and then the extra dimensional volume is infinite. This is because L is a *growing* function at the origin with a positive r-derivative, and then it is "easier" to match the geometry with an exterior L function that also has a positive derivative (and this chooses the plus sign above).

Our matching conditions relate brane parameters with metric deformations in its surroundings, but they do not tell us much about the phenomenological viability of these braneworlds. In an ideal situation, for a given brane energy-momentum and width, and for a given bulk EMT we would impose the matching conditions in the bulk solution (as a perturbation, perhaps, over a known static solution) and find out the implications for the brane induced metric, $g_{\mu\nu}(x,\epsilon)$. The relation between the brane induced metric and its EMT would determine what type of gravity brane observers would feel. But this approach

[6] For the global vortex the exterior geometry is $AdS_5 \times S^1$ [25], but this case is not a solution for a pure cosmological constant in the bulk.

is usually very difficult to implement in practice, since it is very hard to find analytic bulk solutions (even perturbative ones, see [17] for work in this direction). It is however possible to obtain a good deal of information on the curvature of the brane induced metric without actually solving the bulk equations. The idea, that we will elaborate on in more detail in the next section for the cosmological case, is to evaluate Einstein equations just outside the brane. In particular, the rr component of the Einstein tensor, evaluated at $r=\epsilon$ reads,

$$\begin{aligned} G^r_r|_\epsilon &= -\frac{1}{2}R(g) + \frac{M_b^2}{2\pi M_*^4}[\hat{T}^r_r + \hat{T}^\theta_\theta] + \\ &+ \frac{1}{32\pi^2}\frac{M_b^2}{M_*^8}[4\hat{T}^\rho_\sigma\hat{T}^\sigma_\rho - (\hat{T}^\rho_\rho)^2 - 2\hat{T}^\rho_\rho(\hat{T}^r_r+\hat{T}^\theta_\theta) - (\hat{T}^r_r+\hat{T}^\theta_\theta)(5\hat{T}^r_r - 3\hat{T}^\theta_\theta)] = \frac{T^r_r|_\epsilon}{M_*^4}\,, \end{aligned} \tag{2.28}$$

where we have used the matching conditions eqs. (2.19), (2.20) and have neglected the term $M_b\nabla^\rho\nabla_\rho L|_\epsilon$. It is not clear *a priori* that this term is negligible as compared with the induced metric curvature (the integrals of both terms have been neglected in the matching conditions). However it is in general related to $\mu-$derivatives of β that, through our matching conditions, can be argued to be negligible under the assumptions we are using. Again, we will be more specific about the size of the different terms when discussing cosmological solutions. This equation determines the curvature for the induced metric in terms of the brane EMT and the bulk EMT evaluated at the brane boundary. This equation will become much more illuminating when particularised to the cosmological case as we will see in the next section.

3. Cosmology on a codimension 2 brane

In order to study the cosmological implications of our matching conditions we particularise the metric ansatz to

$$ds^2 = N(t,r)^2dt^2 - A(t,r)^2d\vec{x}^2 - dr^2 - L(t,r)^2d\theta^2\,, \tag{3.1}$$

where we have taken flat spatial sections in the brane for simplicity and we set $N(t,0)=1$ by performing a redefinition of the t coordinate. In this section we will assume that eq. (2.17) holds in our system, so we can use the simpler version of the matching conditions, eqs. (2.19), (2.20), instead of the more general one, eqs. (2.8), (2.10). Remember that, since we are assuming that the first derivatives are small compared with the brane width we also have, at the level of approximation we are working, $N(t,\epsilon)\simeq N(t,0)=1$, and also $A(t,0)\simeq A(t,\epsilon)\equiv a(t)$. So the matching conditions, eqs. (2.19), (2.20), take now the form

$$\frac{A'|_\epsilon}{a} = -\frac{M_b}{8\pi M_*^4}[\rho+p-p_r-p_\theta]\,, \tag{3.2}$$

$$N'|_\epsilon = \frac{M_b}{8\pi M_*^4}[3(\rho+p)+p_r+p_\theta]\,, \tag{3.3}$$

$$(1-\beta) = \frac{1}{8\pi M_*^4}[\rho-3p-p_r+3p_\theta]\,, \tag{3.4}$$

where we have taken a "cosmological" brane EMT: $\hat{T}^M_N = \mathrm{diag}(\rho,-p,-p,-p,-p_r,-p_\theta)$. These equations, however, do not yield any information about the cosmology one can expect in these models. One constraint on it is given by the brane EMT conservation, eq. (2.16), that reads for our cosmological set up,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) - \partial_t p_r - (p_r + p_\theta)\frac{\partial_t M_b}{M_b} = 2\pi \frac{T_{0r}^{\text{bulk}}|_\epsilon}{M_b}\,. \tag{3.5}$$

However, we would like to obtain the equations that govern the evolution of the scale factor of the brane, $a(t)$, in terms of the brane EMT. As we said at the end of the previous section, with our matching conditions we could, in principle, find the bulk solution for a given model (as a perturbation, perhaps, over a known static solution) and figure out the implications of this perturbed solution for the time dependence of the scale factor. Instead of doing that we will show that a lot of information on the cosmology of codimension two branes can be obtained by evaluating Einstein equations just outside the brane, following the same approach as [12] for the codimension one case. The matching conditions tell us what the first $r-$derivatives of the metric are at the brane boundary, eqs.(3.2-3.4), whereas second $r-$derivatives can only be found by solving Einstein equations. The crucial point to note is that, out of the five non-zero (for the cosmological metric) components of the Einstein tensor, the tt, xx and $\theta\theta$ involve second r-derivatives and therefore allow us to algebraically compute their values at the brane boundary whereas the tr and the rr components do not involve second $r-$derivatives but only known first $r-$derivatives and first and second time derivatives of the metric. The former gives the brane EMT conservation, eq. (3.5) whereas the latter, evaluated at $r = \epsilon$, reads

$$\begin{aligned} G^r_r|_\epsilon &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) - \frac{M_b^2}{2\pi M_*^4}(p_r + p_\theta) + \\ &+ \frac{1}{32\pi^2}\frac{M_b^2}{M_*^8}\left[3(\rho+p)^2 + (p_r + p_\theta)[2(\rho - 3p) - 5p_r + 3p_\theta]\right] = \frac{T^r_r|_\epsilon}{M_*^4}\,, \end{aligned} \tag{3.6}$$

where we have particularised eq. (2.28) to the cosmological case. Here we can be a bit more specific about the size of the term we are neglecting, which is (we do not explicitly write factors of order one)

$$M_b\, \partial_t^2 L = M_b \int_0^\epsilon dr\, \partial_t^2 L' \approx M_b\, \epsilon\, \partial_t^2 L'|_\epsilon \approx \frac{\partial_t^2 \rho}{M_*^4}\,, \tag{3.7}$$

where in the third equality we have approximated (as a conservative order of magnitude) $\partial_t^2 L'(r \leq \epsilon) \approx \partial_t^2 L'|_\epsilon$, and in the fourth one we have used the matching conditions. We therefore see that, at least at late cosmological times it is utterly negligible as compared with the terms we are keeping. Nevertheless it should be noted that this is just an order of magnitude estimation and one should carefully check this approximation when dealing with particular models (for which this term could play an important role in the cosmology of our brane). The equation we have obtained is a generalised Friedmann equation that incorporates the matching conditions for our general codimension two brane. Taken together with the energy-momentum conservation equation, eq. (3.5), suffices to determine the evolution of the scale factor and therefore the cosmology. As a first check to this equation we can note that we recover the expected behaviour in the case of a pure tension brane with no rr and $\theta\theta$ components of the EMT. This case corresponds to $\rho + p = p_r = p_\theta = 0$

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and we get that the expansion of the brane depends solely on the bulk EMT and not on the brane tension, agreeing with the solutions presented in the literature for this case [13]. As a matter of fact, we could have guessed the generic form of the modified Friedmann equation based on the well defined infinitesimal limit for a pure tension brane as something like

$$3\left(\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}\right)=\mathcal{F}(\rho+p)+\mathcal{G}(p_r,p_\theta,\rho,p)+\frac{T^r_r|_\epsilon}{M_*^4}\,, \tag{3.8}$$

where $\mathcal{F},\mathcal{G}$ are arbitrary functions with the only restriction that $\mathcal{G}(0,0,\rho,p) = 0$ and $\mathcal{F}(0) = 0$. The bulk EMT might also have some implicit dependence on ρ, p and $p_{r,\theta}$. A critical point when considering self-tuning issues is the particular form of the function $\mathcal{F}$. If it was linear in $\rho+p$ it could represent an important step towards a self-tuning scenario yielding a cosmology of the type studied in [29] whereas the quadratic dependence we have actually found would be in conflict with phenomenology. But this negative conclusion in the self-tuning issue is of course a bit premature, and should not be taken too seriously, in the sense that we have not included the possible dependence of the bulk EMT on the brane parameters. A realistic model might exist that has self-tuning features hidden in such dependence. For the time being however we will naively assume that the bulk EMT does not have any implicit ρ, p or $p_{r,\theta}$ dependence, and we have just a cosmological constant in the bulk. In that case, it is possible to obtain a realistic cosmology if one is ready to give up self-tuning considerations. In our universe, the term in this equation proportional to $(\rho+p)^2$ would be extremely small ($\sim \rho^2_{\text{matter}}$), while (assuming p_r and p_θ are constant) the terms that go like p_r+p_θ or $(p_r+p_\theta)(5p_r-3p_\theta)$ would act as a cosmological constant. One expects then that the term $\propto (p_r+p_\theta)(\rho-3p)$ would give the dominant time-dependence and therefore a conventional cosmology

$$3\left(\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}\right)\approx -\frac{M_b^2}{16\pi^2 M_*^8}(p_r+p_\theta)(\rho-3p)+\frac{M_b^2}{2\pi M_*^4}(p_r+p_\theta)+\frac{\Lambda_r}{M_*^4}\,, \tag{3.9}$$

where we have just neglected the terms proportional to $(\rho+p)^2$ and $p^2_{r,\theta}$ and defined $T^r_r|_\epsilon \equiv \Lambda_r$. There remains, of course, the issue of the effective cosmological constant that would have to be tuned to zero. We have several parameters that can be chosen at will in the equation above, so one can fine-tune the effective cosmological constant to a small value by requiring

$$\Lambda_r \simeq -M_b^2\frac{p_r+p_\theta}{2\pi}\left(1-\frac{T_0}{2\pi M_*^4}\right), \tag{3.10}$$

where we have considered that $\rho \sim -p \sim T_0$. There is *a priori* no reason to expect such a cancellation, so this scenario does not seem to yield any light on the cosmological constant problem. But notice that the value of p_θ and p_r coming just from the integration of the bulk cosmological constant inside the brane has an order of magnitude that, for a weakly gravitating brane ($T_0/M_*^4 \ll 1$), already agrees with this requirement: in case of having no "brane" contributions to these quantities one would expect $p_r, p_\theta \sim -2\pi\Lambda_r\epsilon^2 \sim -2\pi\Lambda_r/M_b^2$.

It is very interesting that such a simple model can yield a realistic cosmology, but one would expect that in more involved models the bulk deformation produced by the brane EMT would affect T^r_r also. Depending on how T^r_r reacts under a deformed solution (satisfying our ρ- and p-dependent matching conditions), one obtains different cosmologies. We see that considering a constant $T^r_r = \Lambda_r$, p_r and p_θ can yield a conventional cosmology (with the cosmological constant problem included), but it would be very interesting to see what happens in other cases. In the case of having two compact extra dimensions stabilised with a magnetic flux for instance, the deficit angle in the transverse space affects the local energy density of the flux, and therefore $T^r_r|_\epsilon$ in eq. (3.6), and this could be the dominant effect when determining the cosmology [17]. A deeper study of these issues is currently under way. Nonetheless, it is interesting to point out that in the case of having just a cosmological constant in the bulk, the effective Planck mass is

$$M_{Pl}^2 = -8\pi^2 \frac{M_*^8}{M_b^2}(p_r + p_\theta)^{-1} \simeq 4\pi \frac{M_*^8}{\Lambda_r}\left(1 - \frac{T_0}{2\pi M_*^4}\right), \tag{3.11}$$

where we have used eq. (3.10). Any relation with the extra-dimensional volume is not transparent in this equation but let's imagine that we are in a situation whose exterior metric is approximately described by the geometry given by eq. (2.25), simply AdS space with a compact dimension, and the matching conditions reduce to eqs. (2.26), (2.27). Taking the minus sign in (2.25), so the extra dimensional volume is finite, we can write

$$V_2 \simeq 2\pi \int_0^\infty L_0 e^{-5kr/2} dr = \frac{4\pi L_0}{5k}\,. \tag{3.12}$$

We can use now the matching conditions and the relation $k = \sqrt{-\frac{5\Lambda_r}{2M_*^4}}$ to get

$$M_{Pl}^2 = \frac{25}{4} M_*^4 V_2\,. \tag{3.13}$$

We need to chose the negative sign in the metric (2.25) in order to get a positive Planck mass squared. If we want to match our brane with an exterior AdS geometry as in [26], and obtain a realistic cosmology, we need that $T_0 > 2\pi M_*^4$ (that do not necessarily imply curvatures of order $\sim M_*^2$, as we have previously commented). If on the other hand we insist on a weakly gravitating brane, $T_0 \ll M_*^4$, then our matching conditions show that the appropriate branch is the plus sign in (2.25) and so the volume of the extra dimensions is infinite. Interestingly enough, the effective 4D Planck mass remains finite although imaginary, a situation identical to the codimension one case with a negative tension brane [11]. Also, as we will see below, the dependence on the different scales with high exponents makes it very easy to generate a hierarchically large Planck mass without departing from our approximations. One can point out again several similarities between this possibility and the well known codimension one braneworlds consisting on a 3-brane moving in a 5D AdS space [11, 12]. In both cases the brane EMT enters in the definition of the effective Planck mass, and in both cases we need a non-zero “brane vacuum energy” (the brane tension in the codimension one case, or the $p_r + p_\theta$ parameter in our case) in order to find a realistic cosmology. This is a result of the quadratic dependence of the generalized Friedman equation on the brane EMT parameters in both cases. Also, in both cases we need to fine tune the bulk cosmological constant against brane parameters in order to obtain a small effective cosmological constant on the brane. It is also thanks to this fine-tuning that we recover the relation of the effective 4D Planck mass with the higher dimensional one times the extra-dimensional volume, as expected from KK arguments (as in the codimension one case). It would be interesting to push this analogy further, and we might be able to interpret our thick codimension two brane as a "curled up" codimension one brane, since as we have commented previously, the wall defining the brane boundary is indeed a codimension one hypersurface.

We can also check now the magnitude of the terms we are neglecting in the Einstein equations when doing the matching for these solutions. In order to do so, we consider as an example particular values of the different parameters, motivated by a "TeV brane". We consider a weakly gravitating brane situation, and in this case the exterior geometry would not be of the type given by eq. (2.25), since we need a positive bulk cosmological constant in order to obtain a realistic cosmology (see eq. (3.11)). Taking for instance the brane parameters to be of the order

$$M_b \approx \text{TeV}\,, \tag{3.14}$$

$$\rho^{1/4} \approx (-p)^{1/4} \approx 10\,\text{TeV}\,, \tag{3.15}$$

$$(-p_r)^{1/4} \approx (-p_\theta)^{1/4} \approx 100\,\text{GeV}\,, \tag{3.16}$$

and the fundamental scale

$$M_* \approx 1.7 \times 10^3\ \text{TeV}\,, \tag{3.17}$$

so that the Planck mass comes in the right size

$$M_{Pl} \approx 10^{18}\,\text{GeV}\,,$$

we see that the terms we neglected in the matching are of order

$$\int_0^\epsilon dr\ LR(g) \sim \frac{3}{M_b^2}\left(\frac{\dot{A}^2}{A^2}+\frac{\ddot{A}}{A}\right) \sim 10^{-90}\,, \tag{3.18}$$

$$\int_0^\epsilon dr\ \partial_t^2 L \sim \frac{1}{M_*^4 M_b^2}\partial_t^2(-\rho+3p-p_r+3p_\theta) \sim 10^{-165}\,. \tag{3.19}$$

We have assumed that the rr component of the bulk EMT is fine-tuned to give a small (realistic) Hubble parameter, $\Lambda_r^{1/6} \sim 200\,\text{GeV}$ and for the second equation we have used arguments similar to the ones leading to eq. (3.7). It is therefore clear that these terms are indeed smaller than the ones we have considered, that are of order

$$\int_0^\epsilon dr L'' = 1-\beta \simeq \frac{1}{4M_*^4}(\rho-3p+p_r-3p_\theta) \sim 10^{-9}\,, \tag{3.20}$$

$$\int_0^\epsilon dr(LK)' = \frac{K|_\epsilon}{M_b} \sim -\frac{2(p_r+p_\theta)}{M_*^4} \sim 10^{-17}\,. \tag{3.21}$$

Note that these terms, although much larger than the neglected ones, still have a hierarchy between themselves, as prescribed by our requirement, eq. (2.17). As we said we need the small cosmological constant fine-tuning so that the contributions given by (3.18) are smaller than the ones coming from the terms like (3.21). One can understand the hierarchy of the different terms noticing that the terms we neglected in the matching are in these solutions proportional to the brane EMT squared or its time derivatives (assuming that the term linear in $p_r + p_\theta$ in eq. (3.9) is cancelled by Λ_r), while the terms we have kept are matched with the brane EMT linearly. One should, however, check for any particular solution the level of approximation that using our matching conditions represent, since it is not guaranteed in general that they constitute a good approximation. In case they do not, our matching conditions have corrections that depend on the internal structure of the brane. A numerical estimation of these terms would be advisable when using our matching conditions, although as we have seen one can easily find models with solutions in which they are utterly negligible, and therefore in these models the use of our matching conditions is fully justified.

Before finishing this section we would like to stress that this particular example (and its associated cosmological constant fine-tuning) has nothing to do with a possible self-tuning mechanism. In these models, solutions in which the brane is curved exist on equal footing with solutions in which the brane is flat. So the required tuning should be provided by extra considerations, the most promising being supersymmetry (see [30] and references there in). The crucial point is that a thick brane, via our matching conditions, allows for a dynamical deficit angle and therefore the possibility of self-tuning is present in such a set-up. A more detailed study of self-tuning issues in these models requires knowledge of the bulk solution in more involved models and is currently under way.

4. Conclusions, open issues and future prospects

The goal of this letter was to study the dynamics of the induced metric on a codimension two braneworld, and in particular we were interested in its relation with the brane EMT. The solution is singular in general for an infinitesimally thin brane,[7] except for the case of a pure tension brane. Contrary to what might seem, this case is not relevant for the study of the so called self-tuning properties on codimension two braneworlds due to the staticity (for the deficit angle) of the solution as opposed to the intrinsically dynamical nature of self-tuning. Due to these reasons we had to abandon the thin brane idealization and consider a brane of finite thickness. Our first step was to find the matching conditions in the second section, i.e., the set of equations that relate the brane EMT with the deformation of the surrounding spacetime it produces. Since we are dealing with a thick brane that has a singular thin limit, it was not clear that one should be able to do this without knowing all the precise small-scale structure of the brane. We have however shown that one can obtain this set of equations depending only on the integral of the brane EMT as a good approximation provided certain conditions are met. These conditions have the interpretation of requiring that the brane has an energy-momentum laying mainly along the parallel directions (so it is close to a pure tension brane), with small 4D curvature. We have also seen in this second section that the μr component of the Einstein equations

[7]As we have previously mentioned, this is not the case when one considers Einstein-Gauss-Bonnet gravity on the bulk [16], and in fact one can find non-singular solutions even for infinitesimally thin higher codimension braneworlds when one uses the general Lanczos-Lovelock lagrangians in higher dimensions [22].

gives, when evaluated just outside the brane using our matching conditions, the energy-momentum conservation equation for the brane. Using our matching conditions we have been able to interpret some singular solutions of 6D supergravity found recently in [14] as being sourced by a codimension two defect with certain energy-momentum and we have also paid particular attention to the simple case of having just a cosmological constant in the bulk. In this case we have matched our brane with an exterior AdS geometry as in [26], obtaining the required brane properties.

In the third section, attempting to obtain further information that allows us to assess the phenomenological viability of these models, we have specialized our equations to the cosmological case. The matching conditions relate however the brane EMT with the first derivatives of the metric with respect to r, the orthogonal coordinate, and to obtain the cosmology we would like to relate the brane EMT with the parallel (time) derivatives of the metric. Fortunately, this can be done simply by using the matching conditions in the rr component of the Einstein equation evaluated just outside the brane, at $r = \epsilon$. The reason is that this component of the full 6D Einstein equations is the only one (apart from the μr) that does not involve second $r-$derivatives, so we can get rid of the first ones using our matching conditions and we are left with only time derivatives of the induced metric and brane parameters, obtaining the modified Friedmann equation we were looking for. This procedure is completely analogous to the one followed in the codimension one case by Binetruy *et al.* in [12], the only difference being the added complication here of having to consider a thick brane, since the thin limit is singular for the codimension two case. We have identified a model that could yield a realistic cosmology, simply considering a constant value for the orthogonal components of the brane EMT and fine-tuning the bulk cosmological constant to get a small effective cosmological constant for the brane metric. Put like that, this model does not seem to shed any light on the possible self-tuning behaviour of codimension two brane worlds, one of the original motivations for considering these class of models. It would be interesting in any case to explore this simple model with just a cosmological constant in the bulk in more detail, in particular to study which bulk geometries are obtained from it when we have a positive cosmological constant in the bulk, that as we saw, makes compatible a weakly gravitating brane with a realistic cosmology. In case of having a negative bulk cosmological constant, the exterior geometry is in some cases just AdS space with a compact spatial dimension [26], and for a pure tension brane the bulk is simply a wick rotation and analytical continuation of the AdS-Schwarzschild geometry (see below), so one might expect that the solution asymptotes to AdS space in general when we have a negative cosmological constant in the bulk.

But in fact the main uncertainty, and source of model dependence in our cosmological equations is the bulk EMT: our modified Friedmann equation depends on its rr component. One can expect that in particular models the $\hat{T}_{MN}-$dependent matching conditions for the brane imply that this bulk EMT component also gets an implicit $\hat{T}_{MN}$ dependence, i.e. $T^r_r|_\epsilon$ would also depend on ρ, p, p_r and p_θ for the cosmological case. It is certainly conceivable that this dependence is the dominant one, and one could then obtain different cosmologies depending on the particular model, or compactification, one is dealing with. A more careful examination of these issues for different models will be deferred to a future publication. It is worth pointing out that the bulk and the brane curvatures are independent parameters that in principle have no reason to be related, even for some given brane parameters. This can be seen by considering the 6D black hole with cosmological constant [31], substituting in it $t \to i\theta$ and analytically continuing the constant (r, θ) hypersurfaces to a lorentzian manifold of curvature H^2. Then one can interpret the solution as a pure tension infinitesimally thin codimension two braneworld where H^2 is the curvature of the brane induced metric [32]. One can see then that the brane curvature, the bulk cosmological constant (that determines curvature of the bulk) and the brane tension are independent parameters of the solution. So these models do not have any *selftuning* behaviour *per se*, if certain models only admit flat 4D geometries [14], supersymmetry is to blame, and supergravities with these properties are also known in 7D [33] (where there are 3 compact dimensions and 4 flat dimensions). The problem, of course, is to obtain an effective 4D theory with supersymmetry broken at a high scale without spoiling 4D flatness, and codimension two branes could help in this [30]. It would be very interesting to impose our matching conditions in the solutions of the 6D supergravity that have been proposed in order to realize the selftuning behaviour [14], and in particular to study the implications of these matching conditions for the supersymmetry of the background, but as we said before, a closer examination of these issues is beyond the scope of the present paper, and will be deferred for future work.

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